

On graded polynomial identities with an antiautomorphism

K.I. Beidar, T.-S. Chen, Y. Fong, and W.-F. Ke*

Department of Mathematics, National Cheng Kung University Tainan, Taiwan 701, ROC

Received 7 September 2001

Communicated by Susan Montgomery

Abstract

Let G be a commutative monoid with cancellation and let \mathcal{R} be a strongly G -graded associative algebra with finite G -grading and with antiautomorphism. Suppose that \mathcal{R} satisfies a graded polynomial identity with antiautomorphism. We show that \mathcal{R} is a PI algebra. © 2002 Elsevier Science (USA). All rights reserved.

1. Introduction

Throughout this paper all rings and algebras are associative. The reader is referred to [7,16] for basic concepts and results on rings with (generalized) polynomial identities. Let G be a monoid with unity e and cancellation. Let \mathcal{F} be a commutative ring with 1, and \mathcal{R} an \mathcal{F} -algebra. We say that \mathcal{R} is *almost G -graded* if there are \mathcal{F} -submodules $\mathcal{R}_g \subseteq \mathcal{R}$, $g \in G$, such that $\mathcal{R} = \sum_{g \in G} \mathcal{R}_g$ and $\mathcal{R}_g \mathcal{R}_h \subseteq \mathcal{R}_{gh}$ for all $g, h \in G$. If $\sum_{g \in G} \mathcal{R}_g$ is direct (i.e., $\sum_{g \in G} \mathcal{R}_g = \bigoplus_{g \in G} \mathcal{R}_g$), then we say that \mathcal{R} is *G -graded*. Further, set $\text{supp}(\mathcal{R}) = \{g \in G \mid \mathcal{R}_g \neq 0\}$. The G -grading is said to be *finite* if $|\text{supp}(\mathcal{R})| < \infty$. A G -graded algebra \mathcal{R} is called *strongly G -graded* if

- (1) $\text{supp}(\mathcal{R})$ consists of invertible elements,
- (2) \mathcal{R} has an identity 1, and

* Corresponding author.

(3) $1 \in \mathcal{R}_g \mathcal{R}_{g^{-1}} = \mathcal{R}_e$ for all $g \in \text{supp}(\mathcal{R})$.

When G is the group of order 2, a G -graded algebra is called a *superalgebra*.

Let $U(\mathcal{F})$ be the group of invertible elements of \mathcal{F} , and let \mathcal{R} be a G -graded algebra. Assume that G is commutative. An automorphism $\phi: \mathcal{R} \rightarrow \mathcal{R}$ of the \mathcal{F} -module \mathcal{R} is called an *antiautomorphism of the G -graded algebra \mathcal{R}* if $\mathcal{R}_g^\phi = \mathcal{R}_g$ for all $g \in G$ and there exists a map $\nu: G \times G \rightarrow U(\mathcal{F})$ such that $\nu(e, p) = 1 = \nu(p, e)$ and $(ab)^\phi = \nu(p, q)b^\phi a^\phi$ for all $a \in \mathcal{R}_p, b \in \mathcal{R}_q$, and $p, q \in G$. In the case when \mathcal{R} is a superalgebra with $G = \{e, g\}$ and $\nu(g, g) = -1$, the antiautomorphism ϕ is called a *superinvolution* provided that $\phi^2 = 1$.

Throughout the rest of the paper, we assume the following conditions:

- (1) G is a commutative monoid with cancellation,
- (2) \mathcal{F} is an associative ring,
- (3) \mathcal{R} is an associative \mathcal{F} -algebra with a finite G -grading, and
- (4) $\phi: \mathcal{R} \rightarrow \mathcal{R}$ is an antiautomorphism of the G -graded algebra \mathcal{R} .

Let $X = \bigcup_{g \in G} X_g$ be a disjoint union of infinite sets $X_g, g \in G$, and let $\mathcal{F}\langle X \rangle$ be the free \mathcal{F} -algebra on X . Let \mathcal{A} be an almost G -graded \mathcal{F} -algebra. An element $f(x_1, x_2, \dots, x_n) \in \mathcal{F}\langle X \rangle$ is said to be a *G -graded polynomial identity on \mathcal{A}* provided that $\psi(f) = 0$ for all algebra homomorphisms $\psi: \mathcal{F}\langle X \rangle \rightarrow \mathcal{A}$ with $\psi(X_g) \subseteq \mathcal{A}_g$ for all $g \in G$.

We denote the set $\{x^\phi \mid x \in X\}$ as X^ϕ , and define a map $\delta: X \cup X^\phi \rightarrow G$ by the rule $\delta(x) = g = \delta(x^\phi)$ for all $x \in X_g, g \in G$. Next, given a monomial $M = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_n^{\varepsilon_n} \in \mathcal{F}\langle X \cup X^\phi \rangle$, where each $\varepsilon_i \in \{1, \phi\}$, we set $\delta(M) = \delta(x_1)\delta(x_2) \dots \delta(x_n)$. An element $f(x_1, x_2, \dots, x_n) \in \mathcal{F}\langle X \cup X^\phi \rangle$ is said to be a *G -graded polynomial identity with ϕ on \mathcal{R}* provided that $\psi(f) = 0$ for all algebra homomorphisms $\psi: \mathcal{F}\langle X \cup X^\phi \rangle \rightarrow \mathcal{R}$ with $\psi(X_g) \subseteq \mathcal{R}_g$ and $\psi(x^\phi) = \psi(x)^\phi$ for all $x \in X_g, g \in G$.

Let $h(x_1, x_2, \dots, x_n) \in \mathcal{F}\langle X \cup X^\phi \rangle$ with at least one of its coefficients is equal to 1. It is easy to see that if h is a G -graded polynomial identity with antiautomorphism for \mathcal{R} , then \mathcal{R} satisfies a multilinear G -graded polynomial identity $f(x_1, x_2, \dots, x_n) \in \mathcal{F}\langle X \cup X^\phi \rangle$ with at least one of the coefficients of f is 1. In this case, we may assume, without loss of generality, that the monomial $x_1 x_2 \dots x_n$ is involved in f with 1 as the coefficient, and that there exists $g \in G$ such that $\delta(N) = g$ for all monomials N involved in f . We set

$$G_f = \{\delta(x_1), \delta(x_2), \dots, \delta(x_n)\} \subseteq G.$$

In 1986 Bergen and Cohen [8] proved that \mathcal{R} is PI provided that G is a finite group, \mathcal{F} is a field, and \mathcal{R}_e is a PI algebra. This result was extended to algebras over arbitrary commutative rings by Kelarev [11]. Bahturin and Zaicev [3] obtained an analogous result for algebras over a field with finite G -grading where G is any monoid with cancellation. Sehgal and Zaicev [17] proved that if H

is a normal subgroup of a group G with finite index and the group algebra $\mathcal{F}[G]$, considered as G/H -graded algebra, satisfies a G/H -graded polynomial identity, then $\mathcal{F}[G]$ is a PI algebra. Note, that in this case $\mathcal{F}[G]$ is a strongly G/H -graded algebra. Recently Beidar and Chebotar obtained the following generalization of their result.

Theorem 1.1 [5, Theorem 1.1]. *Let G be a monoid with unity e and cancellation, let \mathcal{F} be a commutative ring with 1, and let \mathcal{R} be an almost G -graded \mathcal{F} -algebra with finite G -grading satisfying a G -graded multilinear polynomial identity $f(x_1, x_2, \dots, x_n)$. Suppose that the monomial $x_1x_2 \dots x_n$ is involved in f with coefficient 1, $\delta(N) = \delta(x_1x_2 \dots x_n)$ for all monomials N involved in f and $G_f \subseteq \text{supp}(\mathcal{R})$. Then:*

- (i) *If \mathcal{R} is a prime ring and $|\text{supp}(\mathcal{R})| = 2$, then the ring \mathcal{R}_e contains a nonzero ideal satisfying the standard identity St_{2n-2} of degree $2n - 2$, and the ring \mathcal{R} satisfies a nontrivial generalized polynomial identity. If in addition \mathcal{R} is a simple ring with 1, then \mathcal{R} is a PI algebra.*
- (ii) *If both \mathcal{R} and \mathcal{R}_e are prime rings, then \mathcal{R}_e satisfies St_{2n-2} and \mathcal{R} is a PI algebra.*
- (iii) *If \mathcal{R} has an identity $1 \in \mathcal{R}_e$, G_f consists of invertible elements, and $\mathcal{R}_g\mathcal{R}_{g^{-1}} = \mathcal{R}_e$ for all $g \in G_f$, then \mathcal{R} is a PI algebra.*

On the other hand, in 1969 Amitsur [2] proved that a ring \mathcal{A} satisfying a polynomial identity with involution is PI (see [1,9,13] for earlier results). Motivated by the aforesaid results we prove the following theorem.

Theorem 1.2. *Let G be a commutative monoid with unity e and cancellation, let \mathcal{F} be a commutative ring with 1, and let \mathcal{R} be a G -graded \mathcal{F} -algebra with an antiautomorphism ϕ . Suppose that $|\text{supp}(\mathcal{R})| < \infty$, and that \mathcal{R} satisfies a G -graded multilinear polynomial identity $f(x_1, x_2, \dots, x_n)$ with antiautomorphism such that the monomial $x_1x_2 \dots x_n$ is involved in f with coefficient 1, $\delta(N) = \delta(x_1x_2 \dots x_n)$ for all monomials N involved in f , and $G_f \subseteq \text{supp}(\mathcal{R})$. Then:*

- (i) *If \mathcal{R} is a prime ring and $|\text{supp}(\mathcal{R})| = 2$, then the ring \mathcal{R}_e contains a nonzero ideal satisfying the standard identity St_{4n-2} of degree $4n - 2$, and the ring \mathcal{R} satisfies a nontrivial generalized polynomial identity. If in addition \mathcal{R} is a simple ring with 1, then \mathcal{R} is a PI algebra.*
- (ii) *If \mathcal{R} and \mathcal{R}_e are both prime rings, then \mathcal{R}_e satisfies St_{4n-2} and \mathcal{R} is a PI algebra.*
- (iii) *If \mathcal{R} has an identity $1 \in \mathcal{R}_e$, G_f consists of invertible elements, and $\mathcal{R}_g\mathcal{R}_{g^{-1}} = \mathcal{R}_e$ for all $g \in G_f$, then \mathcal{R} is a PI algebra.*

We now give the following examples to justify the necessity of the conditions set in Theorem 1.2. These examples are modification of Examples 1–3 from [5].

Example 1.3. Let $G = \langle a \rangle$ be a cyclic group of order 3. There exists a G -graded algebra \mathcal{R} over a field with an antiautomorphism ϕ such that \mathcal{R} is a simple Artinian ring not satisfying a (generalized) polynomial identity, \mathcal{R}_e is a direct sum of two skew fields and \mathcal{R} satisfies a G -graded polynomial identity with antiautomorphism $f(x, y) = xy^\phi$, $x, y \in \mathcal{R}_a$, such that $G_f \subseteq \text{supp}(\mathcal{R})$.

Indeed, let \mathcal{D} be a skew field with an antiautomorphism ψ which is not a PI ring (for instance, \mathcal{D} may be the classical ring of quotients of the Weyl algebra A_1 over the rational number field with involution $x_1^\psi = y_1$ and $y_1^\psi = -x_1$ [15]). Let $\mathcal{F} = Z(\mathcal{D})$ be the center of \mathcal{D} , let $\mathcal{R} = M_2(\mathcal{D})$ be the \mathcal{F} -algebra of 2×2 matrices over \mathcal{D} and let $\{e_{ij} \mid 1 \leq i, j \leq 2\}$ be a system of matrix units of \mathcal{R} . Further, set $u = e_{11}$, $v = e_{22}$, and

$$\mathcal{R}_e = u\mathcal{R}u + v\mathcal{R}v, \quad \mathcal{R}_a = u\mathcal{R}v \quad \text{and} \quad \mathcal{R}_{a^2} = v\mathcal{R}u.$$

Define an antiautomorphism ϕ of \mathcal{R} by the rule

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\phi = \begin{pmatrix} d^\psi & -b^\psi \\ -c^\psi & a^\psi \end{pmatrix} \quad \text{for all } a, b, c, d \in \mathcal{D}$$

and note that \mathcal{R} is a G -graded algebra satisfying a G -graded polynomial identity with antiautomorphism $f(x, y) = xy^\phi$, $x, y \in \mathcal{R}_a$.

Hence the first statement of the theorem does not hold in general if $|\text{supp}(\mathcal{R})| = 3$. Next, the second statement does not hold in general if \mathcal{R}_e is not prime even if \mathcal{R} is a simple Artinian ring and \mathcal{R}_e is a direct sum of two skew fields.

Example 1.4. Let $G = \{e, g\}$ be a cyclic group of order 2. There exists a G -graded algebra \mathcal{R} over a field with an antiautomorphism $*$ such that \mathcal{R} is a simple ring (without identity) satisfying a generalized polynomial identity, \mathcal{R} is not a PI algebra, \mathcal{R} satisfies a G -graded polynomial identity

$$f(x_1, \dots, x_5) = [x_1, x_2]x_3[x_4, x_5], \quad x_1, x_2, x_4, x_5 \in \mathcal{R}_e, \quad x_3 \in \mathcal{R}_g,$$

and $G_f = \text{supp}(\mathcal{R}) = G$ (see Theorem 1.2(i)).

Indeed, let \mathcal{F} be a field, let \mathcal{R} be the \mathcal{F} -algebra of infinite matrices with finite number of nonzero entries and let u be the matrix whose $(1, 1)$ entry is equal to 1 and all the other ones are equal to 0. Obviously $uxyuy - uyuxu$ is a generalized polynomial identity on \mathcal{R} and \mathcal{R} is not a PI algebra. Further, set

$$\mathcal{R}_e = u\mathcal{R}u + (1 - u)\mathcal{R}(1 - u) \quad \text{and} \quad \mathcal{R}_g = u\mathcal{R}(1 - u) + (1 - u)\mathcal{R}u.$$

Clearly \mathcal{R} is a G -graded algebra. Next the transpose map $*$ is an antiautomorphism of the G -graded algebra \mathcal{R} and \mathcal{R} satisfies the G -graded polynomial identity $f(x_1, \dots, x_5)$.

Example 1.5. Let $G = \{e, g\}$ be a cyclic group of order 2 and let \mathcal{F} be a field. For any positive integer n the algebra $\mathcal{R} = M_n(\mathcal{F})$ admits a G -grading such that \mathcal{R} is a strongly G -graded algebra with antiautomorphism satisfying the G -graded polynomial identity $f(x_1, \dots, x_5)$ (see Example 2).

Indeed, let $u = e_{11}$. As above set $\mathcal{R}_e = u\mathcal{R}u + (1-u)\mathcal{R}(1-u)$ and $\mathcal{R}_g = u\mathcal{R}(1-u) + (1-u)\mathcal{R}u$. Obviously $\mathcal{R}_g^2 = \mathcal{R}_e$ and so \mathcal{R} is strongly G -graded. We already know that $f(x_1, \dots, x_5)$ is a G -graded polynomial identity on \mathcal{R} and the transpose map is an antiautomorphism of the G -graded algebra \mathcal{R} . On the other hand, the minimal degree of a polynomial identity on \mathcal{R} is $2n$ [16, Lemma 1.4.3]. Therefore there exists no function $m = m(\deg(f))$ such that a simple algebra with 1 satisfying the G -graded polynomial identity f satisfies a polynomial identity of degree m even if \mathcal{R} is a strongly G -graded simple finite-dimensional algebra (see Theorem 1.2(iii)).

The following two corollaries are special cases of the above theorem.

Corollary 1.6. *Let \mathcal{R} be a strongly G -graded algebra with identity and having an antiautomorphism. Suppose that $|\text{supp}(\mathcal{R})| < \infty$, and \mathcal{R} satisfies a G -graded multilinear polynomial identity $f(x_1, x_2, \dots, x_n)$ with antiautomorphism such that the monomial $x_1x_2 \dots x_n$ is involved in f with coefficient 1. Then \mathcal{R} is a PI algebra.*

Corollary 1.7. *Let \mathcal{R} be a superalgebra with superinvolution. Suppose that \mathcal{R} satisfies a graded multilinear polynomial identity $f(x_1, x_2, \dots, x_n)$ with superinvolution such that the monomial $x_1x_2 \dots x_n$ is involved in f with coefficient 1. Further, assume that \mathcal{R} is a prime ring. Then \mathcal{R} satisfies a nonzero generalized polynomial identity. If in addition \mathcal{R} is a simple ring with 1, then \mathcal{R} is a PI algebra.*

We also obtain the following generalization of Amitsur's result [2] on algebras with polynomial identities with involution.

Corollary 1.8. *Let \mathcal{F} be a commutative ring with 1, and \mathcal{R} an \mathcal{F} -algebra with antiautomorphism ϕ . Suppose that \mathcal{R} satisfies a polynomial identity with ϕ , and at least one of the coefficients of the polynomial is equal to 1. Then \mathcal{R} is a PI algebra.*

Proof. Let $f(x_1, x_2, \dots, x_n) \in \mathcal{F}\langle X \cup X^\phi \rangle$ be a multilinear polynomial identity with ϕ on \mathcal{R} such that at least one coefficient of f is equal to 1. Let $\mathcal{R}^\#$ be the ring \mathcal{R} with 1 adjoined. Clearly, $\mathcal{R}^\#$ satisfies $f([x_1, y_1], \dots, [x_n, y_n])$. Set $G = \{e\}$ and $\mathcal{R}_e^\# = \mathcal{R}^\#$. Then $\mathcal{R}^\#$ is a strongly G -graded algebra. The result now follows from Corollary 1.6. \square

2. Proof of main theorem

We first set some further notation in place and obtain some preliminary results for rings.

Let \mathcal{A} be a ring. Given right \mathcal{A} -modules \mathcal{U} and \mathcal{V} and a module map $h : \mathcal{U}_{\mathcal{A}} \rightarrow \mathcal{V}_{\mathcal{A}}$, we denote by hx the image of $x \in \mathcal{U}$ under h . If \mathcal{T} is a nonempty subset of \mathcal{A} , we set

$$\ell(\mathcal{U}; \mathcal{T}) = \{x \in \mathcal{U} \mid x\mathcal{T} = 0\}.$$

Let n be a positive integer and let $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_n, \mathcal{M}$ be right \mathcal{A} -modules. We shall use \mathbf{u}_n to denote the element $(u_1, u_2, \dots, u_n) \in \prod_{k=1}^n \mathcal{L}_k$, and use $\hat{\mathbf{u}}_n^i$ to denote the element

$$(u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n) \in \prod_{\substack{k=1, \\ k \neq i}}^n \mathcal{L}_k \quad \text{for } i \in \{1, 2, \dots, n\}.$$

Let $a \in \mathcal{A}$ be fixed. For nonnegative integers s and t with $t \leq n$, let

$$E_{ij} : \prod_{\substack{k=1, \\ k \neq i}}^{n-t} \mathcal{L}_k \rightarrow \text{Hom}_{\mathcal{A}}(\mathcal{L}_i, \mathcal{M}) \quad (1 \leq i \leq n-t \text{ and } 0 \leq j \leq s+t)$$

be maps having the property that

$$\sum_{i=1}^{n-t} \sum_{j=0}^{s+t} E_{ij}(\hat{\mathbf{u}}_{n-t}^i) u_i a^j = 0 \quad \text{for all } \mathbf{u}_{n-t} \in \prod_{k=1}^{n-t} \mathcal{L}_k.$$

If $E_{ij} = 0$ for all i and j , they certainly have the above property. On the other hand, under certain conditions, the converse is also true.

Lemma 2.1 [5, Lemma 2.1]. *Suppose that the following conditions are satisfied:*

- (i) *For any $0 \leq r \leq n + s - 1$ there exist a positive integer $m = m(r)$ and elements $b_{rq}, c_{rq} \in \mathcal{A}$, $q = 1, 2, \dots, m$, such that*

$$d_r = \sum_{q=1}^m b_{rq} a^r c_{rq} \neq 0 \quad \text{and}$$

$$\mathcal{M} \sum_{q=1}^m b_{rq} a^p c_{rq} = 0 \quad \text{for all } p = 0, 1, \dots, n + s - 1, p \neq r.$$

- (ii) $\ell(\mathcal{M}; \mathcal{A}d_r) = 0$ for all $r = 0, 1, \dots, n + s - 1$.

Then $E_{ij} = 0$ for all i and j .

Now let ϕ be an antiautomorphism of \mathcal{A} , and assume that $\mathcal{L}_k, 1 \leq k \leq n$, are \mathcal{A} - \mathcal{A} -bimodules with the unary operation ϕ such that $\phi : \mathcal{L}_k \rightarrow \mathcal{L}_k$ is a bijective map, $(xc)^\phi = c^\phi x^\phi$ and $(cx)^\phi = x^\phi c^\phi$ for all $c \in \mathcal{A}$ and $x \in \mathcal{L}_k, 1 \leq k \leq n$. Further, let $J \subseteq \{1, 2, \dots, n\}$ with $|J| = n - t$ and let

$$F_{ip} : \prod_{\substack{k=1, \\ k \neq i}}^n \mathcal{L}_k \rightarrow \text{Hom}_{\mathcal{A}}(\mathcal{L}_i, \mathcal{M}) \quad (1 \leq i \leq n \text{ and } 0 \leq p \leq t),$$

$$E_{jq} : \prod_{\substack{k=1, \\ k \neq j}}^n \mathcal{L}_k \rightarrow \text{Hom}_{\mathcal{A}}(\mathcal{L}_i, \mathcal{M}) \quad (j \in J \text{ and } 0 \leq q \leq t)$$

be maps such that

$$\sum_{i=1}^n \sum_{p=0}^t F_{ip}(\hat{u}_n^i) u_i a^p + \sum_{j \in J} \sum_{q=0}^t E_{jq}(\hat{u}_n^j) u_j^\phi a^q = 0 \tag{1}$$

for all $u_n \in \prod_{k=1}^n \mathcal{L}_k$.

The following result, which we shall need in the sequel, is a generalization of both Lemma 2.1 and [6, Theorem 3.3].

Lemma 2.2. *Suppose that the following conditions are satisfied:*

- (i) *For any $0 \leq r \leq 2n - 1$ there exist a positive integer $m = m(r)$ and elements $b_{rk}, c_{rk} \in \mathcal{A}, k = 1, 2, \dots, m$, such that*

$$d_r = \sum_{k=1}^m b_{rk} a^r c_{rk} \neq 0 \quad \text{and}$$

$$\mathcal{M} \sum_{k=1}^m b_{rk} a^s c_{rk} = 0 \quad \text{for all } s = 0, 1, \dots, 2n - 1 \text{ with } s \neq r,$$

- (ii) $\ell(\mathcal{M}; Ad_r) = 0$ for all $r = 0, 1, \dots, 2n - 1$.

Then $F_{ip} = 0$ and $E_{jq} = 0$ for all i, j, p , and q .

Proof. We proceed by induction on $n - t$. If $n - t = 0$, then (1) reads

$$\sum_{i=1}^n \sum_{p=0}^n F_{ip}(\hat{u}_n^i) u_i a^p = 0 \quad \text{for all } u_n \in \prod_{k=1}^n \mathcal{L}_k,$$

and the result follows from Lemma 2.1.

In the inductive case $n - t > 0$, we may assume without loss of generality that $n \in J$. Set $J' = J \setminus \{n\}$. Substituting $a^{\phi^{-1}} u_n$ for u_n in (1), and using the

notation $(\hat{u}_{n-1}^i, a^{\phi^{-1}} u_n)$ for $(u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_{n-1}, a^{\phi^{-1}} u_n)$ and the likes, we obtain

$$\sum_{i=1}^{n-1} \sum_{p=0}^t F_{ip}(\hat{u}_{n-1}^i, a^{\phi^{-1}} u_n) u_i a^p + \sum_{p=0}^t F_{np}(\mathbf{u}_{n-1}) a^{\phi^{-1}} u_n a^p + \sum_{j \in J'} \sum_{q=0}^t E_{jq}(\hat{u}_{n-1}^j, a^{\phi^{-1}} u_n) u_j^\phi a^q + \sum_{q=0}^t E_{nq}(\mathbf{u}_{n-1}) u_n^\phi a^{q+1} = 0 \quad (2)$$

for all $\mathbf{u}_n \in \prod_{k=1}^n \mathcal{L}_k$. Multiplying (1) by a from the right and subtracting the resulting expression from (2), we see that

$$\sum_{i=1}^n \sum_{p=0}^{t+1} \tilde{F}_{ip}(\hat{u}_n^i) u_i a^p + \sum_{j \in J'} \sum_{q=0}^{t+1} \tilde{E}_{jq}(\hat{u}_n^j) u_j^\phi a^q = 0 \quad (3)$$

for all $\mathbf{u}_n \in \prod_{k=1}^n \mathcal{L}_k$, where

$$\tilde{F}_{i0}(\hat{u}_n^i) = F_{i0}(\hat{u}_{n-1}^i, a^{\phi^{-1}} u_n), \quad 1 \leq i \leq n-1,$$

$$\tilde{F}_{ip}(\hat{u}_n^i) = F_{ip}(\hat{u}_{n-1}^i, a^{\phi^{-1}} u_n) - F_{i,p-1}(\hat{u}_n^i), \quad 1 \leq i \leq n-1, \quad 1 \leq p \leq t, \quad (4)$$

$$\tilde{F}_{i,t+1}(\hat{u}_n^i) = -F_{it}(\hat{u}_n^i), \quad 1 \leq i \leq n-1, \quad (5)$$

$$\tilde{F}_{n0}(\hat{u}_n^n) = F_{n0}(\hat{u}_n^n) a^{\phi^{-1}},$$

$$\tilde{F}_{np}(\hat{u}_n^n) = F_{np}(\hat{u}_n^n) a^{\phi^{-1}} - F_{n,p-1}(\hat{u}_n^n), \quad 1 \leq p \leq t, \quad (6)$$

$$\tilde{F}_{n,t+1}(\hat{u}_n^n) = -F_{nt}(\hat{u}_n^n), \quad (7)$$

and the maps \tilde{E}_{jq} ($j \in J$ and $0 \leq q \leq t+1$) are defined similarly. Applying the induction assumption on (3), we see that, in particular, $\tilde{F}_{ip} = 0$ for $1 \leq i \leq n$ and $0 \leq p \leq t+1$.

Now, (5) implies that $F_{it} = 0$ for $1 \leq i \leq n-1$. From (4) we infer that $F_{ip} = 0$ for all $1 \leq i \leq n-1$ and $0 \leq p \leq t$. Analogously, (6) and (7) together yield that $F_{np} = 0$ for $0 \leq p \leq t$. Taking these into account, the identity (1) becomes

$$\sum_{j \in J} \sum_{q=0}^t E_{jq}(\hat{u}_n^j) u_j^\phi a^q = 0 \quad \text{for all } \mathbf{u}_n \in \prod_{k=1}^n \mathcal{L}_k.$$

Substituting $u_k^{\phi^{-1}}$ for u_k ($k = 1, 2, \dots, n$) in the above equation, we obtain

$$\sum_{j \in J} \sum_{q=0}^t E_{jq}(u_1^{\phi^{-1}}, \dots, u_{j-1}^{\phi^{-1}}, u_{j+1}^{\phi^{-1}}, \dots, u_n^{\phi^{-1}}) u_j a^q = 0,$$

and the result follows at once from Lemma 2.1. \square

As a special case of Lemma 2.2, we have the following corollary.

Corollary 2.3. *Let $F_i, E_i : \prod_{\substack{k=1 \\ k \neq i}}^n \mathcal{L}_k \rightarrow \text{Hom}_{\mathcal{A}}(\mathcal{L}_i, \mathcal{M}), i = 1, 2, \dots, n$, be maps such that*

$$\sum_{i=1}^n F_i(\hat{u}_n^i)u_i + \sum_{i=1}^n E_i(\hat{u}_n^i)u_i^\phi = 0 \quad \text{for all } u_n \in \prod_{k=1}^n \mathcal{L}_k.$$

Suppose that there exists an element $a \in \mathcal{A}$ such that the following conditions are satisfied:

- (i) *There exist a positive integer m and elements $b_{rk}, c_{rk} \in \mathcal{A}, 0 \leq r \leq 2n - 1, 1 \leq k \leq m$, such that for all $r = 0, 1, \dots, 2n - 1$,*

$$d_r = \sum_{k=1}^m b_{rk}a^r c_{rk} \neq 0 \quad \text{and}$$

$$\mathcal{M} \sum_{k=1}^m b_{rk}a^s c_{rk} = 0 \quad \text{for all } s = 0, 1, \dots, 2n - 1 \text{ with } s \neq r.$$

- (ii) $\ell(\mathcal{M}; Ad_r) = 0$ for all $r = 0, 1, \dots, 2n - 1$.

Then $F_i = E_i = 0$ for all i .

Before we can prove Theorem 2.1, some more results about G -graded \mathcal{F} -algebras have to be stated.

Proposition 2.4 [5, Proposition 2.3]. *Let G be a monoid with cancellation and let \mathcal{R} be an almost G -graded algebra with finite G -grading. Let $n = |\text{supp}(\mathcal{R})|$, let m be a positive integer, let $\mathcal{L} = \sum_{g \in G} \mathcal{L}_g$ be a G -graded subring of \mathcal{R} (i.e., $\mathcal{L}_g \subseteq \mathcal{R}_g$ is a subgroup and $\mathcal{L}_g \mathcal{L}_h \subseteq \mathcal{L}_{gh}$ for all $g, h \in G$) and let \mathcal{I} be a right ideal of \mathcal{R}_e . Further, let*

$$H = \{g \in \text{supp}(\mathcal{R}) \mid g \text{ is not invertible in } G \text{ or } \mathcal{R}_g \mathcal{R}_{g^{-1}} = 0\}$$

and let \mathcal{U} be the ideal of \mathcal{R} generated by $\sum_{h \in H} \mathcal{R}_h$. Then:

- (i) *If $(\mathcal{L}_e)^m = 0$, then $\mathcal{L}^{nm} = 0$.*
- (ii) *If $\mathcal{I}^m = 0$, then $(\mathcal{I}\mathcal{R})^{nm} = 0$.*
- (iii) *\mathcal{U} is a nilpotent ideal of \mathcal{R} .*

Proposition 2.5 [5, Proposition 2.4]. *Let G be a monoid with cancellation, \mathcal{R} be an almost G -graded algebra and $n = |\text{supp}(\mathcal{R})| < \infty$. Suppose that \mathcal{R} is a prime algebra. Then \mathcal{R}_e is a semiprime algebra containing nonzero ideals $\mathcal{I}_1, \dots, \mathcal{I}_m$, such that:*

- (a) $\mathcal{I}_i \cap \mathcal{I}_j = 0$ for all $i \neq j$;

- (b) $\mathcal{I} = \bigoplus_{k=1}^m \mathcal{I}_k$ is an essential ideal of \mathcal{R}_e ;
- (c) each $\mathcal{I}_k, k = 1, 2, \dots, m$, is a prime ring;
- (d) $m \leq n$;
- (e) if $d_k \in \mathcal{I}_k \setminus \{0\}$ ($k = 1, 2, \dots, m$) and $d = \sum_{k=1}^m d_k$, then $\ell(\mathcal{R}_g, \mathcal{R}_e d) = 0$ for all $g \in G$.

The next result is a special case of [3, Theorem 3].

Theorem 2.6. *Let G be a monoid with cancellation, let \mathcal{F} be a field and let \mathcal{R} be an almost G -graded algebra with finite G -grading. If \mathcal{R}_e is a PI algebra, then so is \mathcal{R} .*

Theorem 2.7 [12, Theorem 3]. *Let \mathcal{E} be a class of rings which is closed under direct powers and homomorphic images. If every prime ring in \mathcal{E} satisfies a generalized polynomial identity, then \mathcal{E} consists of PI rings.*

Now, we are ready to prove the main theorem.

Proof of Theorem 1.2. Let \mathcal{P} be a prime ideal of \mathcal{R} . Set $\overline{\mathcal{R}} = \mathcal{R}/\mathcal{P}$, and for $g \in G$, set $\overline{\mathcal{R}}_g = (\mathcal{R}_g + \mathcal{P})/\mathcal{P}$. It is clear that $\overline{\mathcal{R}}$ is an almost G -graded \mathcal{F} -algebra. Given $a \in R$, we denote $\bar{a} = a + \mathcal{P} \in \overline{\mathcal{R}}$.

It follows from Proposition 2.4(iii) that $\text{supp}(\overline{\mathcal{R}})$ consists of invertible elements and that

$$\overline{\mathcal{R}}_g \overline{\mathcal{R}}_{g^{-1}} \neq 0 \quad \text{for all } g \in \text{supp}(\overline{\mathcal{R}}). \tag{8}$$

Next, Proposition 2.5 implies that $\overline{\mathcal{R}}_e$ is a semiprime ring. Write

$$f(x_1, \dots, x_n) = f(\mathbf{x}_n) = \sum_{i=1}^n f_i(\hat{\mathbf{x}}_n^i) x_i + \sum_{i=1}^n g_i(\hat{\mathbf{x}}_n^i) x_i^\phi,$$

where all $f_i(\hat{\mathbf{x}}_n^i)$ and $g_i(\hat{\mathbf{x}}_n^i)$ are multilinear polynomials in $x_1, x_1^\phi, \dots, x_{i-1}, x_{i-1}^\phi, x_{i+1}, x_{i+1}^\phi, \dots, x_n, x_n^\phi$. Let $\mathcal{I}_1, \mathcal{I}_2, \dots, \mathcal{I}_w$ be ideals of $\overline{\mathcal{R}}_e$ as in Proposition 2.5. Assume that each $\mathcal{I}_l, l = 1, 2, \dots, w$, does not satisfy St_{4n-2} . We claim that

$$\overline{f_i(\hat{\mathbf{u}}_n^i) u_i} = 0 \quad \text{for all } i \in \{1, 2, \dots, n\} \text{ and } \mathbf{u}_n \in \prod_{k=1}^n \mathcal{R}_{\delta(x_k)}. \tag{9}$$

Indeed, fix $1 \leq l \leq w$ and recall that \mathcal{I}_l is a prime ring. Let \mathcal{E}_l be the Martindale (extended) centroid of \mathcal{I}_l . If every elements of \mathcal{I}_l is algebraic of degree $\leq 2n - 1$ over \mathcal{E}_l , then \mathcal{I}_l is a subring of the ring of $(2n - 1) \times (2n - 1)$ matrices over the algebraic closure of \mathcal{E}_l (see [4, p. 3928]), and so Amitsur–Levitzki theorem [16] implies that \mathcal{I}_l satisfies St_{4n-2} , a contradiction. Therefore \mathcal{I}_l contains an element \bar{a}_l which is not algebraic of degree $2n - 1$ over \mathcal{E}_l , which means that

$1, \bar{a}_l, \bar{a}_l^2, \dots, \bar{a}_l^{2n-1}$ are linearly independent over \mathcal{E}_l . By [7, Theorem 2.3.3], we see that for any $r, 0 \leq r \leq 2n - 1$, there exist a positive integer $m = m(l, r)$ and elements $\bar{b}_{lrk}, \bar{c}_{lrk} \in \mathcal{I}_l, k = 1, 2, \dots, m$, such that

$$\begin{aligned} \bar{d}_{lr} &= \sum_{k=1}^m \overline{b_{lrk} a_l^r c_{lrk}} \neq 0 \quad \text{and} \\ \sum_{k=1}^m \overline{b_{lrk} a_l^s c_{lrk}} &= 0 \quad \text{for all } s = 0, 1, \dots, 2n - 1, s \neq r. \end{aligned}$$

We may assume without loss of generality that m does not depend on both l and r . Now, set

$$\begin{aligned} \bar{a} &= \sum_{l=1}^w \bar{a}_l, \\ \bar{b}_{rk} &= \sum_{l=1}^w \bar{b}_{lrk}, \quad \bar{c}_{rk} = \sum_{l=1}^w \bar{c}_{lrk} \quad (1 \leq r \leq 2n - 1 \text{ and } 1 \leq k \leq m), \end{aligned}$$

and put

$$\bar{d}_r = \sum_{k=1}^m \overline{b_{rk} a^r c_{rk}} \quad (1 \leq r \leq 2n - 1).$$

Then we have

$$\begin{aligned} \bar{d}_r &= \sum_{l=1}^m \bar{d}_{lr} \neq 0 \quad \text{and} \\ \sum_{k=1}^m \overline{b_{rk} a^s c_{rk}} &= 0 \quad \text{for } s = 1, 2, \dots, 2n - 1 \text{ with } s \neq r, \end{aligned} \tag{10}$$

for all $r = 1, 2, \dots, 2n - 1$. It follows from Proposition 2.5(e) that

$$\ell(\bar{\mathcal{R}}_g, \bar{\mathcal{R}}_e \bar{d}_r) = 0 \quad \text{for all } g \in G \text{ and } r = 1, 2, \dots, 2n - 1. \tag{11}$$

Without loss of generality, we may assume that $a, b_{rk}, c_{rk}, d_r \in \mathcal{R}_e$ for all r and k . Now, regard each $\bar{\mathcal{R}}_g$ as a right \mathcal{R}_e -module. Then, from (10), we obtain, for $r \in \{1, 2, \dots, 2n - 1\}$ and $g \in G$, that

$$\begin{aligned} d_r &= \sum_{k=1}^m b_{rk} a^r c_{rk} \neq 0 \quad \text{and} \\ \bar{\mathcal{R}}_g \sum_{k=1}^m b_{rk} a^s c_{rk} &= 0 \quad \text{for } s = 1, 2, \dots, 2n - 1 \text{ with } s \neq r. \end{aligned} \tag{12}$$

Moreover, (11) yields

$$\ell(\overline{\mathcal{R}}_g, \mathcal{R}_e d_r) = 0 \quad \text{for all } g \in G \text{ and } r = 1, 2, \dots, 2n - 1. \tag{13}$$

The left multiplications (induced by the right \mathcal{R} -module structure on $\overline{\mathcal{R}}$) by $f_i(\hat{u}_n^i)$ and by $g_i(\hat{u}_n^i)$ ($i = 1, 2, \dots, 2n - 1$ and $u_n \in \mathcal{R}^n$) can be viewed as elements of $\text{Hom}_{\mathcal{R}_e}(\mathcal{R}_{\delta(x_i)}, \overline{\mathcal{R}}_h)$ where $h = \delta(x_1 x_2 \dots x_n)$. Since $f_i(\hat{u}_n^i)u_i = f_i(\hat{u}_n^i)u_i$, we see from Corollary 2.3 that (9) is fulfilled. Note that (9) is true for any multilinear polynomial $f(x_n)$ with antiautomorphism ϕ such that $f(u_n) = 0$ for all $u_n \in \prod_{k=1}^n \mathcal{R}_{\delta(x_k)}$.

(i) Assume that \mathcal{R} is prime and $|\text{supp}(\mathcal{R})| = 2$. Replacing \mathcal{F} by $\mathcal{F}/\ell(\mathcal{F}; \mathcal{R})$ we may assume that \mathcal{F} is an integral domain. Setting $S = \mathcal{F} \setminus \{0\}$ and considering the $S^{-1}\mathcal{F}$ -algebra $S^{-1}\mathcal{R}$, we reduce the proof to the case when \mathcal{F} is a field.

Since \mathcal{R} is prime, Proposition 2.4 implies that $e \in \text{supp}(\mathcal{R})$. Let $g \in \text{supp}(\mathcal{R})$ with $g \neq e$. Recalling that $g^{-1} \in \text{supp}(\mathcal{R})$, we conclude that $g = g^{-1}$ and so $g^2 = e$.

We claim that $\ell(\mathcal{R}_e; \mathcal{R}_g) = \{a \in \mathcal{R}_e \mid a\mathcal{R}_g = 0\} = 0$. Indeed, let $b \in \ell(\mathcal{R}_e; \mathcal{R}_g)$. Then

$$b\mathcal{R}\mathcal{R}_g = b(\mathcal{R}_e\mathcal{R}_g) + (b\mathcal{R}_g)\mathcal{R}_g \subseteq b\mathcal{R}_g = 0$$

and so $b = 0$ because \mathcal{R} is prime and $\mathcal{R}_g \neq 0$.

We now set $h = \delta(x_n) \in G_f \subseteq \text{supp}(\mathcal{R}) = \{e, g\}$. It follows from the above result together with semiprimeness of \mathcal{R}_e that $\ell(\mathcal{R}_e; \mathcal{R}_h) = 0$. Let $\mathcal{K} = \ell(\mathcal{R}; \mathcal{R}_h)$. Clearly \mathcal{K} is a G -graded ring and a left ideal of \mathcal{R} . Next, $\mathcal{K}_e = \ell(\mathcal{R}_e; \mathcal{R}_h) = 0$ and so Proposition 2.4(i) implies that $\mathcal{K}^2 = 0$. As \mathcal{R} is prime, $\mathcal{K} = 0$.

Assume that \mathcal{R}_e has no nonzero ideals satisfying St_{4n-2} . Then (9) (with $\mathcal{P} = 0$) implies that $f_n(u_{n-1}) \in \mathcal{K} = 0$, and so we conclude that $f_n(x_1, \dots, x_{n-1})$ is a G -graded polynomial identity with antiautomorphism on \mathcal{R} . Making use of induction on $n = \deg(f)$, we get that \mathcal{R}_e contains a nonzero ideal satisfying St_{4n-4} and so St_{4n-2} , a contradiction. Therefore \mathcal{R}_e contains a nonzero ideal \mathcal{I} satisfying St_{4n-2} . Now set $\mathcal{L}_p = \mathcal{I}\mathcal{R}_p$, $p \in G$, and $\mathcal{L} = \sum_{p \in G} \mathcal{L}_p$. Since \mathcal{L}_e satisfies St_{4n-2} , Theorem 2.6 implies that \mathcal{L} is a PI algebra. It now follows from [10] that \mathcal{R} satisfies a nonzero generalized polynomial identity (see also [7, Theorem 6.3.20]). Suppose that in addition \mathcal{R} is a simple ring with 1. Then the central closure of \mathcal{R} is equal to \mathcal{R} . It now follows from Martindale theorem on prime rings with generalized polynomial identity [14] that \mathcal{R} has a nonzero socle and the associated skew field is finite-dimensional over its center (see also [7, Theorem 6.1.6]). Since \mathcal{R} is simple, it coincides with its socle. In particular, 1 is an idempotent of finite rank and so Litoff's theorem [7, Theorem 4.3.11] yields that \mathcal{R} is a matrix ring over a skew field which is of finite dimensional over its center. Therefore \mathcal{R} is a PI algebra and the first statement of the theorem is proved.

(ii) Now assume that both \mathcal{R} and \mathcal{R}_e are prime rings. As above we reduce the proof to the case when \mathcal{F} is a field. If \mathcal{R}_e has a nonzero ideal \mathcal{I} satisfying St_{4n-2} ,

then \mathcal{R}_e satisfies St_{4n-2} and so Theorem 2.6 implies that \mathcal{R} is PI. Assume that \mathcal{R}_e has no nonzero ideals satisfying St_{4n-2} . Then (9) implies that $f_n(\mathbf{u}_{n-1})u_n = 0$ for all $\mathbf{u}_n \in \prod_{k=1}^n \mathcal{R}_{\delta(x_k)}$. Setting

$$\mathcal{I} = \mathcal{R}_{\delta(x_n)}\mathcal{R}_{\delta(x_n)-1}, \quad \mathcal{K}_g = \{b \in \mathcal{R}_g \mid b\mathcal{I} = 0\}, \quad \text{and} \quad \mathcal{K} = \sum_{g \in G} \mathcal{K}_g,$$

we see that \mathcal{K} is a left ideal of \mathcal{R} . Since \mathcal{I} is a nonzero ideal of \mathcal{R}_e by (8), we conclude that $\mathcal{K}_e = 0$. Therefore, Proposition 2.4(i) yields that \mathcal{K} is a nilpotent ideal of \mathcal{R} , forcing $\mathcal{K} = 0$. As $f_n(\mathbf{u}_{n-1}) \in \mathcal{K}$, we see that $f_n(x_{n-1})$ is a G -graded polynomial identity on \mathcal{R} . The second statement of the theorem now follows from induction on $\text{deg}(f)$.

(iii) Suppose that $1 \in \mathcal{R}$, G_f consists of invertible elements, and $1 \in \mathcal{R}_g\mathcal{R}_{g-1}$ for all $g \in G_f$.

Let r be a positive integer and let \mathcal{H}_r be the class of all homomorphic images of G -graded algebras \mathcal{B} with finite G -grading, with antiautomorphism, satisfying multilinear G -graded polynomial identity f with antiautomorphism in which the monomial $x_1x_2 \dots x_n$ is involved with coefficient 1 and such that for any $g \in G_f$ there exist $u_1, u_2, \dots, u_r \in \mathcal{B}_g$ and $v_1, v_2, \dots, v_r \in \mathcal{B}_{g-1}$ with $\sum_{i=1}^r u_i v_i = 1$. Clearly the class \mathcal{H}_r is homomorphically closed and is closed under direct powers. Further, $\mathcal{R} \in \mathcal{H}_r$ for some integer r . In view of Theorem 2.7 it is enough to show that every prime homomorphic image $\overline{\mathcal{B}}$ of a G -graded algebra $\mathcal{B} \in \mathcal{H}_r$ satisfies a nonzero generalized polynomial identity.

If $\overline{\mathcal{B}}_e$ contains a nonzero ideal satisfying St_{4n-2} , then as in the proof of (i) we get that $\overline{\mathcal{B}}$ satisfies a nonzero generalized polynomial identity. Therefore we may assume without loss of generality that $\overline{\mathcal{B}}_e$ has no nonzero ideals satisfying St_{4n-2} . Set $g = \delta(x_n)$. It follows from (9) that $\overline{f_n(\mathbf{u}_{n-1})\mathcal{B}_g} = 0$ for all $\mathbf{u}_{n-1} \in \prod_{k=1}^{n-1} \mathcal{B}_{\delta(x_k)}$. Since $\overline{1} \in \overline{\mathcal{B}}_g\overline{\mathcal{B}}_{g-1}$, we conclude that $\overline{f_n(\mathbf{u}_{n-1})} = 0$ for all $\mathbf{u}_{n-1} \in \prod_{k=1}^{n-1} \mathcal{B}_{\delta(x_k)}$. Proceeding inductively on $k = \text{deg}(h)$, where h is a G -graded polynomial with antiautomorphism in which the monomial $x_1x_2 \dots x_m$ is involved with coefficient 1, such that $\overline{h(\mathbf{u}_m)} = 0$ for all $\mathbf{u}_m \in \prod_{k=1}^m \mathcal{B}_{\delta(x_k)}$, we see that $\overline{\mathcal{B}}$ satisfies a nonzero generalized polynomial identity. The proof is thereby complete. \square

Acknowledgment

The authors express their gratitude to the referee for the valuable suggestions.

References

[1] S.A. Amitsur, Rings with involution, *Israel J. Math.* 6 (1968) 99–106.
 [2] S.A. Amitsur, Identities in rings with involution, *Israel J. Math.* 7 (1969) 63–68.

- [3] Yu.A. Bahturin, M.V. Zaicev, Identities of graded algebras, *J. Algebra* 205 (1998) 1–12.
- [4] K.I. Beidar, M.A. Chebotar, On functional identities and d -free subsets of rings, I, *Comm. Algebra* 28 (2000) 2925–2951.
- [5] K.I. Beidar and M.A. Chebotar, When is a graded PI algebra a PI algebra? *Comm. Algebra*, accepted.
- [6] K.I. Beidar, W.S. Martindale 3rd, On functional identities in prime rings with involution, *J. Algebra* 203 (1998) 491–532.
- [7] K.I. Beidar, W.S. Martindale 3rd, A.V. Mikhalev, *Rings with Generalized Identities*, Dekker, 1996.
- [8] J. Bergen, M. Cohen, Actions of commutative Hopf algebras, *Bull. London Math. Soc.* 18 (1986) 159–164.
- [9] I.N. Herstein, Special simple rings with involution, *J. Algebra* 6 (1967) 369–375.
- [10] S.K. Jain, Prime rings having one-sided ideal with polynomial identity coincide with special Johnson rings, *J. Algebra* 19 (1971) 125–130.
- [11] A.V. Kelarev, On semigroup graded PI algebras, *Semigroup Forum* 47 (1993) 294–298.
- [12] M. Kępczyk, E. Puczyłowski, Rings which are sums of two subrings satisfying polynomial identities, *Comm. Algebra* 29 (2001) 2059–2065.
- [13] W.S. Martindale 3rd, Rings with involution and polynomial identities, *J. Algebra* 11 (1969) 186–194.
- [14] W.S. Martindale 3rd, Prime rings satisfying a generalized polynomial identity, *J. Algebra* 12 (1969) 576–584.
- [15] J.C. McConnell, J.C. Robson, *Noncommutative Noetherian Rings*, Wiley, 1987.
- [16] L.H. Rowen, *Polynomial Identities in Ring Theory*, Academic Press, 1980.
- [17] S.K. Sehgal, M.V. Zaicev, Graded identities of group algebras, *Comm. Algebra* 30 (2002) 489–505.